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A REVIEW OF DEVELOPMENTS IN LATTICE GAUGE THEORY: EXTREME ENVIRONMENTS AND DUALITY TRANSFORMATIONS

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ABSTRACT

Recent calculations showing that various lattice gauge theories lose their quark confining property abruptly at a finite, large temperature are reviewed. Models of confinement at finite baryon density are discussed. Duality transformations and the phase diagrams of Abelian gauge theories in ordinary environments are reviewed. The use of exact and approximate duality transformations on gauge theories in general is stressed as a useful new tool.

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INTRODUCTION

I would like to review some recent work on lattice gauge theories. First, we discuss the possibility that these theories lose their quark confining property when placed in extreme environments such as high temperature[1,2] or density.[3] The second topic reviews duality transformations of Abelian and Z_N lattice gauge theories which relate them to more familiar physical systems whose qualitative properties are known. For example, Abelian lattice gauge theories can be mapped onto the theory of interacting closed threads[4] which is in turn related (approximately) to conventional scalar electrodynamics.[5]

The first topic, confinement in extreme environments, may have astrophysical implications. The large baryon density in some neutron stars and the high temperatures present in the early stages of the universe may be sufficiently extreme environments so that strongly interacting matter is better described as quarkium than as nuclear matter.[6] Recent work on lattice gauge theories indicate that as the temperature T is increased, a critical point T_c is reached where confinement is abruptly lost.[1,2] In an Abelian lattice theory with no quarks the potential energy V(R) of an external static quark antiquark pair whose members are a distance R apart changes from linear,

$$V(R) = \text{const.}\ |R|, \qquad T < T_c \tag{1a}$$

to Coulombic,

$$V(R) = Q^2/|R|, \qquad T > T_c \tag{1b}$$

above the critical point. The phase transition at T_c is second order. An even more interesting phenomenon occurs in non-Abelian lattice gauge theories of pure gluonic matter: below T_c Eq. (1a) applies, but above the transition the force law become short-ranged,

$$V(R) \sim e^{-\mu|R|}, \qquad T > T_c \tag{2}$$

How is it possible that quarks which are in the fundamental representation of the gauge group can be screened by gluons which lie in the octet representation? This behavior is analogous to placing a charge of arbitrary strength in a plasma and finding that there is no residue of Coulomb's law. Apparently the charge fluctuations of the medium are sufficient to screen any impurity, even its charge is incommensurate with the fundamental charges in the medium. So, Eq. (2) teaches us that the non-Abelian gluon medium forms a non-Abelian plasma which can screen quarks of any color.

One would guess that the critical temperature is on the order of $kT_c \approx 1$ GeV, a typical hadron mass. If so, this phase transition might have only been relevant in the earliest stages of the creation of the universe. Extreme environments of a less remote variety may occur in the interiors of neutron stars--here the baryon density is suspected to be several times that of ordinary nuclear matter. In this environment the nucleons are overlapping considerably so one is led to suspect that quarks are more relevant to a description of the system than protons and neutrons. To address this question one would like to place QCD into an environment of variable baryon density b and search for qualitative changes in the theory's character as b ranges from zero upward. Such a study has been made in 1+1 dimensional models of confinement and a rich spectrum of phenomena have been found.[3] The $q\bar{q}$ potential weakens continuously as b increases in some of these models, i.e. without an abrupt phase transition. Systematic studies of lattice gauge theories in 3+1 dimensions in the presence of a background baryon density have not been carried out.

In this talk we emphasize lattice gauge theories and some models of confinement. But the questions posed here have been studied in continuum QCD as well. It has been argued that at high temperature and/or high density many properties of these theories can be analyzed using renormalization-group-improved perturbation theory. The reason for this is, roughly, that high temperature or density bring a large momentum scale into the problem (through large kinetic energies $\sim kT$, or a huge Fermi energy) about which renormalization-group-improved perturbation theory can be performed. Since these theories are asymptotically free such calculations are reliable. Among the results obtained in this way is the computation of the equation of state (free quarks plus calculable corrections) for QCD in a large baryon density and a derivation of the plasma phase of QCD at high temperature. The lattice and model calculations add more detail to these results, e.g. the behavior of the theories at all temperatures and densities, precise predictions in the vicinity of phase transitions, etc. In ref. 7 I have attempted to collect a partial list of contributions to the continuum QCD studies. (It is not possible to compile a complete list--I apologize in advance to any author whose work I have carelessly omitted.)

Before discussing these topics in detail, one should realize that the results cited above mean that QCD in extreme environments becomes rather conventional. In ordinary environments QCD presumably confines the quanta created by its fundamental fields. However, at high temperature, for example, this will not be the case. This fact undermines a frequent criticism of QCD which objects to any theory whose fundamental quanta cannot be isolated. The advocates of this view claim that "if the fundamental quanta cannot be isolated then they must be irrelevant .... a more direct, simpler version of strong interactions should be formulated without the excess baggage of quarks and gluons." Since QCD liberates its quarks and gluons at high temperatures, these quanta are manifestly "real" and this objection loses much of its impact.

EXTREME ENVIRONMENTS

Let's consider the argument that Abelian lattice gauge theory undergoes a second order phase transition at finite temperature from a quark confining to an ordinary theory.[1,2] The strategy of the analysis consists in relating this model to a simpler, more familiar spin lattice whose phases are known. Then the calculation of the $q\bar{q}$ potential is mapped into the calculation of the spin-spin correlation function and the results of Eq. (1) and (2) are read off. The correspondences are:

$$\begin{aligned} &\text{Lattice Gauge Theory Partition Function} \Leftrightarrow \text{3-dimensional XY Model Partition Function} \\ &\text{Temperature} \Leftrightarrow \text{Inverse Temperature} \\ &q\bar{q}\ \text{Potential} \Leftrightarrow -\ell n[\text{Spin-Spin Correlation Function}] \end{aligned} \tag{3}$$

These correspondences establish a duality relation between the two models. Since it is known that the 3-dimensional XY model has a second order phase transition at T*, the correspondences imply the existence of a phase transition in the Abelian lattice gauge theory at a temperature T_c. At a temperature in the XY model above T*, the spin-spin correlation function $\langle s(R)s(0)\rangle$ decays exponentially with R. Using the correspondences, this means that the lattice gauge theory confines for $T < T_c$,

$$\begin{aligned} V(R) &\sim -\ell n \langle s(R)s(0)\rangle \\ &\sim -\ell n\, e^{-\mu|R|} \\ &\sim \mu|R| \end{aligned} \tag{4}$$

Similarly, at a temperature below T* in the XY model the spin-spin correlation function approaches a constant, the magnetization squared, at a rate determined by spin wave analysis, $\langle s(R)s(0)\rangle \sim m^2 e^{-c/|R|}$, as $R \to \infty$. This means that for $T > T_c$ in the lattice gauge theory,

$$V(R) \sim -\ln \langle s(R)s(0)\rangle$$
$$\sim -\ln(m\, e^{-c/|R|})$$
$$\sim c/|R| + \text{const.} \qquad (5)$$

which reproduces Coulomb's law!

Let's establish the first part of the duality relations. The Partition Function for Abelian lattice gauge theory at finite $\beta = 1/kT$ is

$$Z(\beta) = \sum_{\substack{\text{physical}\\ \text{states}}} e^{-\beta\cdot(\text{energy of state})} = \operatorname*{Tr}_{\substack{\text{physical}\\ \text{states}}} e^{-\beta H} \qquad (6)$$

where H is the Hamiltonian of the theory (time is continuous and there are 3 discrete spatial axes: The lattice spacing is "a"). The restriction of the sum in Eq. (6) to "physical states" is important. To appreciate it recall the construction of the Hamiltonian form of the theory.[8] Space is discrete with <u>sites</u> labelled r, a triplet of integers, and directed <u>links</u> $(r,\hat{n})$, which begin on site r and point one lattice unit in the $\hat{n}$ direction. The degrees of freedom of the theory consist of phases

$$e^{i\phi(r,\hat{n})} \qquad (7)$$

defined on links, and there are variables $E(r,\hat{n})$ conjugate to the angular variables $\phi(r,\hat{n})$,

$$[\phi(r,\hat{n}),E(r',\hat{n}')] = i\delta_{r,r'}\delta_{n,n'} \qquad (8)$$

(ϕ is related to the vector potential $\underset{\sim}{A}$ of QED, $\phi(r,\hat{n}) \to g\underset{\sim}{A}(r)\cdot\hat{n}a$, and E is the electric flux operator, $E(r,\hat{n}) \to \underset{\sim}{E}(r)\cdot\hat{n}a^2$. One can check that Eq. (8) reproduces the canonical commutation relations between $\underset{\sim}{A}$ and the electric field $\underset{\sim}{E}$ of QED formulated in the $A_0 = 0$ gauge when the continuum limit $a \to 0$ is taken.)

The Hamiltonian of the lattice theory consists of two pieces: a lattice analog of the electric field squared, and a lattice analog of the magnetic field squared. The first term is easily constructed since $E(r,\hat{n})$ is the operator which measures the electric flux passing from site to site,

$$\frac{1}{2}\int \underset{\sim}{E}^2\, d\underset{\sim}{x} \to \frac{g^2}{2a}\sum_{\text{links}} E^2(r,\hat{n}) \qquad (9)$$

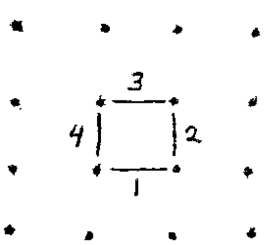

Fig. 1
A plaquette

An interesting lattice form of the square of the magnetic field makes use of the phase variables introduced in Eq. (7). Consider four links 1,2, 3, and 4 which form a closed square, Fig. 1. Associate with each square the product of phases,

$$\exp i(\phi(1) + \phi(2) + \phi(3) + \phi(4))$$

Then a possible correspondence is[9,10]

$$\frac{1}{2}\int \underset{\sim}{B}^2\, d\underset{\sim}{x} \rightarrow -\frac{2}{ag^2} \sum_{\text{squares}} \cos(\phi(1)+\phi(2)+\phi(3)+\phi(4)) \quad (10)$$

(One can check that this correspondence becomes an equality in the classical continuum limit. To do this, observe that Stokes' law and the relation $\phi(r,\hat{n}) \rightarrow g\underset{\sim}{A}(r)\cdot\hat{n}a$ imply that the sum of ϕ's around a square is proportional to the magnetic flux passing through the square, and that if ϕ is a smooth field and the $a \rightarrow 0$ limit is taken only the quadratic term in the cos term survives in Eq. (10).) The lattice Hamiltonian now reads[8]

$$H = \frac{g^2}{2a} \sum_{\text{links}} E^2(r,\hat{n}) - \frac{2}{ag^2} \sum_{\text{squares}} \cos(\phi(1) + \ldots + \phi(4)) \quad (11)$$

The important features of this expression are that it has exact gauge invariance for any lattice spacing, and that ϕ is effectively bounded, $-\pi < \phi < \pi$.

This formulation of the theory is fully specified once the physical space of states is defined. Recall that the quantization of electromagnetism in the class of gauges $A_0 = 0$ requires a subsidiary condition. It reads

$$\sum_{\hat{n}} E(r,\hat{n})|\text{physical state}\rangle = 0 \quad (12)$$

which is a discrete form of Gauss' law. This constraint and the gauge condition reduce the number of independent degrees of freedom of the gauge field down to two. Eq. (12) allows us to picture physical states as closed loops of electric flux constructed so that electric flux is conserved at each site. Furthermore, since $E(r,\hat{n})$ is conjugate to an angular variable $\phi(r,\hat{n})$ the spectrum of $E(r,\hat{n})$ consists of just the integers--electric flux is quantized in this theory. So, the flux a link can be 0, ±1, ±2, etc. This allows the physical states to be enumerated in a countable fashion. Some examples are shown in Fig. 2.

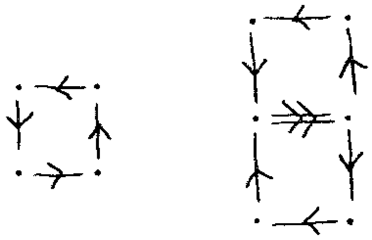

Fig. 2 Physical states

Now we can turn back to the expression for the Partition function, Eq. (6). The sum over physical states can be replaced by a constrained sum over all states

$$Z(\beta) = \sum_{\substack{\text{all}\\ \text{states}}} \prod_{r} \delta_{\nabla\cdot E(r),0}\, \epsilon^{-\beta(\text{energy of state})} \quad (13)$$

The Kronecker symbols enforce the constraints of Eq. (12). Now let's simplify the problem of computing $Z(\beta)$: replace the Hamiltonian by just the electric flux term,

$$H \rightarrow \frac{g^2}{2a} \sum_{\text{links}} E^2(r,\hat{n}) \quad (14)$$

This replacement is justified because the electric term in H guarantees confinement at $T = 0$. Using Eq. (14) we shall still find that the theory loses confinement at a finite T_c. Therefore, the inclusion of the magnetic term, which weakens the confining potential at $T = 0$, can only reduce T_c--it could not change the argument that a finite T_c exists. Recall how the electric term in H leads to confinement. Place a static quark at position $r = 0$ on the lattice and a static antiquark at position $r = R$. To be physical this state must satisfy Gauss' law. This means that a unit of electric flux must connect the $q\bar{q}$ pair. The lowest energy configuration of flux is given by the shortest path between the pair, Fig. 3. But according to Eq. (14) each link on this path costs an energy $g^2/2a$ and there are R/a links, so the total energy is $g^2/2a^2 \cdot |R|$--the linear confining potential.

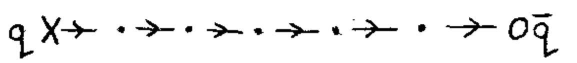

Fig. 3 Static quarks and flux

At this stage the Partition function is

$$Z(\beta) = \sum_{\substack{\text{all}\\ \text{states}}} \prod_r \delta_{\nabla\cdot E(r),0} \exp\left(-\beta \frac{g^2}{2a} \sum_{\text{links}} E^2(r,\hat{n})\right) \quad (15)$$

This is the Partition function for a three-dimensional system of closed ($\nabla\cdot E = 0$) threads. The aficionado of spin systems will recognize Eq. (15) as the partition function of the XY model in three dimensions[4,11]--the correspondences in Eq. (3) are immediate to him and he is done. Let's work out these connections.

Consider a three-dimensional lattice whose sites are occupied by two dimensional spins,

$$s(r) = \begin{pmatrix}\cos\theta(r)\\ \sin\theta(r)\end{pmatrix} \quad (16)$$

The Hamiltonian consists of nearest neighbor ferromagnetic couplings. In terms of $\theta(r)$ we have

$$H = J \sum_{r,\mu} (1 - \cos\Delta_\mu\theta(r)) \quad (17)$$

in an abbreviated but suggestive notation in which Δ_μ is a discrete difference operator ($\Delta_\mu f(r) \equiv f(r+\hat{\mu}a) - f(r)$) in the direction μ ($\mu=1,2,3$) between sites. The statistical mechanics of this model follows from the Partition function,

$$Z(\beta) = \prod_r \int_{-\pi}^{\pi} \frac{d\theta(r)}{2\pi} \exp\left(-\beta \sum_{r,\mu} (1 - \cos\Delta_\mu\theta(r))\right) \quad (18)$$

This expression is difficult to work with so we will replace it by a simpler model which has been analyzed in detail. To motivate it, we note that the integrand of $Z(\beta)$ is a periodic function of $\theta(r)$. Therefore, it can be written as a Fourier series. For large β (small temperature) the Fourier transform of $\exp(\cos\Delta_\mu\theta(r))$ is well approximated by a Gaussian,

$$\exp(\beta\cos\Delta_\mu\theta) \rightarrow \sum_{\ell_\mu=-\infty}^{\infty} \exp(+i\ell_\mu\Delta_\mu\theta)\exp(-\ell_\mu^2/4\beta) \tag{19}$$

The right-hand side of this replacement preserves all the important features of the original spin lattice and is numerically precise for low temperature. But the right-hand side is well-defined for all β and is more easily analyzed because of its Gaussian form. It is referred to as the Villain model[12] and has been studied in its own right. We now concentrate on it and show that its partition function is dual to the Abelian lattice gauge theory at finite temperature. Making the replacement Eq. (19) in the Partition function Eq. (18), we see that each $\theta(r)$ integral can be done and that each generates a familiar constraint, $\Delta_\mu\ell_\mu(r) = 0$. Now, Eq. (18) becomes

$$Z = \sum_{\ell_\mu(r)=-\infty}^{\infty} \prod_r \delta_{\Delta_\mu\ell_\mu(r),0} \exp(-1/4\beta \sum_{r,\mu} \ell_\mu^2(r)) \tag{20}$$

Comparing Eq. (15) with Eq. (20) we have established the first part of the duality relation: Z(lattice model) $\rightarrow$ Z(spin system) if T(lattice model) $\rightarrow$ 1/T (spin system). Since the Villain model is known to have a second order phase transition at $T_c \leq 6.2$,[13] we learn that the lattice model also has two phases. We shall now see that the phases can be labelled by different qualitative behaviors of the $q\bar{q}$ potential and that the potential is dual to the spin-spin correlation function of the Villain model.

To calculate the $q\bar{q}$ potential we must compute the Partition function with a source of one unit of flux at r = 0 and a sink of one unit of flux R lattice sites away. Now the expression for Z is the same as in Eq. (15) except the constraint at r = 0 reads $\delta_{\nabla\cdot E,1}$ and that at r = R reads $\delta_{\nabla\cdot E,-1}$.

$$Z^{q\bar{q}} = \sum_{\substack{\text{all}\\ \text{states}}} \prod_r \delta_{\nabla\cdot E(r),Q(r)} \exp\left[-\beta\frac{g^2}{2a}\sum_{\text{links}} E^2(r,n)\right] \tag{21}$$

where $Q(r) = \delta_{r,R} - \delta_{r,0}$ is the charge density of the external quarks. The inter-quark potential is the free energy of this system minus the free energy of the theory without the external quarks,

$$V(R) = -\frac{1}{\beta}[\ln Z^{q\bar{q}} - \ln Z] = -\frac{1}{\beta}\ln(Z^{q\bar{q}}/Z) \tag{22}$$

Keeping this result in mind we now turn to the spin-spin correlation function in the Villain model. Consider the function,

$$\begin{aligned} C(R) &= \langle s_1(R) + is_2(R), s_1(0) + is_2(0)\rangle \\ &= \langle e^{i[\theta(R)-\theta(0)]}\rangle \end{aligned} \tag{23}$$

In the spin system Eq. (18) C(R) becomes

$$C(R) = Z(R)/Z \tag{24a}$$

where

$$Z(R) \equiv \prod_r \int_{-\pi}^{\pi} \frac{d\theta(r)}{2\pi} \exp\left\{-\beta \sum_{r,\mu} (1-\cos\Delta_\mu\theta(r)) + i(\theta(R)-\theta(0))\right\} \tag{24b}$$

Making the Villain replacement, Z(R) becomes more simply

$$Z(R) = \prod_r \int_{-\pi}^{\pi} \frac{d\theta(r)}{2\pi} \sum_{\ell_\mu(r)=-\infty}^{\infty} \exp\left\{-\frac{1}{4\beta}\sum_{r,\mu} \ell_\mu^2(r)\right\} \exp\left\{i\sum_{r,\mu} \ell_\mu(r)\Delta_\mu\theta(r)\right\}$$
$$\times \exp\left\{i[\theta(R)-\theta(0)]\right\} \tag{25}$$

All the integrations over $\theta(r)$ can be done as before except at the special sites $r = 0$ and $r = R$. The additional factor $\exp(i\theta(R))$ at the site R replaces the constraint by $\Delta_\mu\ell_\mu = 1$, and the factor $\exp(-i\theta(0))$ at the site 0 produces the constraint $\Delta_\mu\ell_\mu = -1$. So, we have

$$Z(R) = \sum_{\ell_\mu(r)=-\infty}^{\infty} \prod_r \delta_{\Delta_\mu\ell_\mu(r),Q(r)} \exp\left\{-\frac{1}{4\beta}\sum_{r,\mu} \ell_\mu^2(r)\right\} \tag{26}$$

where the "charge" $Q(r) = \delta_{r,R}-\delta_{r,0}$. Thus Z(R) is just the partition for the lattice gauge theory in the presence of a static quark at R and an antiquark at $r = 0$! So, now we have the last line in the Duality relations of Eq. (3),

$$V(R) \sim -\ln C(R) \tag{27}$$

where the temperature of the spin system maps onto inverse temperature of the lattice gauge theory. The force laws then discussed in the Introduction follow and we are done.

The arguments for the more interesting non-Abelian theory are similar--the gauge theory computations are mapped onto properties of a non-Abelian spin system in an external field. This system's phase diagram is also well known. The results of this analysis have been discussed above.

Now let's turn briefly to environments at ordinary temperatures but large baryon density ρ. Since the Fermi surface is pushed to high energy as ρ increases, one would guess that many features of the theory resemble the theory of free quarks. Is there a phase transition at a critical density ρ_c at which the theory passes from one of confinement to something qualitatively different? This question has not been answered in lattice gauge theory. However, several models of confinement have been studied in detail.[3] One model was the two species massive Schwinger model-QED in 1 time-1 space dimensions with massive fermions. One species is given an (Abelian) color charge +g and the other -g. The spectrum of this theory consists of "hadrons" --colorless bound states of fermions--which interact locally via an interaction density whose strength is proportional to the fermion mass m. An environment which is colorless but rich in baryon number can be constructed and the force law between a static $q\bar{q}$ pair studied in the usual way. For large baryon density ρ, the potential behaves as

$$V(R) \sim \frac{\text{const.}}{\rho} \cdot \sin^2(\pi \frac{e}{g}) \cdot |R| \qquad (28)$$

where e is the color charge of the static quarks. The strength of the linear potential falls to zero smoothly as ρ increases indicating the absence of a phase transition. Of course, the equation of state at high ρ is given by that of 2 species of free quarks plus small, calculable corrections. It would be interesting to calculate some dynamical properties of the 2 species Schwinger model at finite baryon density ρ to see the interplay between the "hadron" character and "quark" character of the theory. Of course, one dimensional models of neutron stars may not be particularly good guides to the real world (!), but studies beyond renormalization-group-improved perturbation theory have not been done in QCD and lattice gauge theories have not been considered at high density.

PHASES OF ABELIAN GAUGE THEORIES

The last topic I would like to describe is (unfortunately) also esoteric. Its aim is to understand the phases of Abelian and Z_N lattice gauge theories[14] by relating them to other more familiar models. This work was motivated by the fact that Abelian lattice gauge theory confines for strong coupling. However, for weak coupling it reduces to conventional continuum QED. Therefore, somewhere in the intermediate coupling region a phase transition (or something similar) must occur to separate these two qualitatively different behaviors. Ideally one wants to find an expression for the Partition function of the model which yields a useful physical picture of the critical region. One approach to this problem takes advantage of the recent analyses of Abelian spin systems alluded to in the previous section. In particular, a similar conceptual problem existed a few years ago in the 2 dimensional XY model. At high temperature general theorems assure us the system is disordered and the correlation function of a spin at r = 0 and one at r = R falls exponentially with R. However, high temperature expansions indicated that the theory's susceptibility diverges at a non-zero temperature.[15] In addition, low temperature analyses indicated that spin waves are the only relevant excitations in the system.[12] But spin waves produce a spin-spin correlation function which vanishes as a temperature-dependent power of R. The challenge then was to find the excitations responsible for the qualitative change in the spin-spin correlation function in the critical region. Those excitations turned out

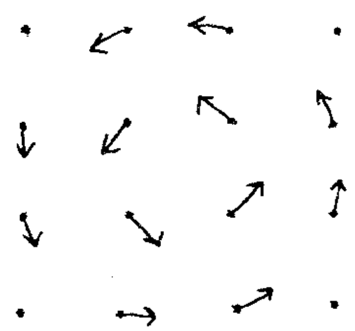

Fig. 4 A vortex

to be vortices--spin configurations which, roughly speaking, have non-zero winding number[16] (Fig. 4). These vortices effect the spins over all of space and clearly tend to disorder the system. A famous energy vs. energy argument[16] indicates that vortices are only important in the statistical mechanics of the system above a critical temperature T_c. Using the Villain approximation and a renormalization group analysis[17], these ideas have been placed on relatively sound footing and an appealing physical picture of the phases of the model has been obtained: For $T < T_c$, only spin wave excitations are important and the spin-spin correlation function is power-behaved. The vortices are bound into small vortex-antivortex pairs and cannot disorder the system. This is the low temperature "dielectric" phase of the model. As T approaches T_c from below the entropy of the system increases until the vortex pairs become unbound (T_c is an ionization point) and the vortices become free. In this "conducting phase" the vortices disorder the system completely.

This bag of tricks can be played on the Abelian lattice gauge theory. Beginning with the space-time symmetric version of the theory[10] (space and time are discrete), the Partition function,

$$Z = \prod_{r,\hat{n}} \int_{-\pi}^{\pi} \frac{d\phi(r,\hat{n})}{2\pi} \exp\left[-\frac{1}{g^2} \sum_{r,\hat{\ell},\hat{n}} [1-\cos(\nabla_{\hat{\ell}}\phi(r,\hat{n})-\Delta_{\hat{n}}\phi(r,\hat{\ell}))]\right] \tag{29a}$$

can be written as[4]

$$Z = \prod_{r,\mu} \sum_{m_\mu(r)=-\infty}^{\infty} \prod_{r} \delta_{\Delta_\mu m_\mu(r),0} \exp\left[-\frac{\pi}{g^2} m_\mu(r)V(r-r')m_\mu(r')\right] \tag{29b}$$

In Eq. (29a) the unit vector $\hat{n}$ ranges over the four directions of the links in the space-time lattice. The index μ serves the same purpose in Eq. (29b). In that expression V is the four dimensional Coulomb potential. So, Eq. (29b) has changed the problem of understanding the Abelian lattice gauge theory written in terms of the lattice version of $F_{\mu\nu}$ into the problem of understanding a four dimensional system of closed current loops interacting through a Coulomb potential, Fig. 5. These loops are analogous to the vortices of the XY model. The spin-spin correlation is analogous to the expectation value of Wilson's line integral[10] $\exp\left[i \sum_{\text{closed contour}} \phi(r,\hat{n})\right]$. The reader should recall that this line integral is the path integral expression equivalent to the computation of the static $q\bar{q}$ potential discussed earlier. In fact, choosing a closed rectangular contour

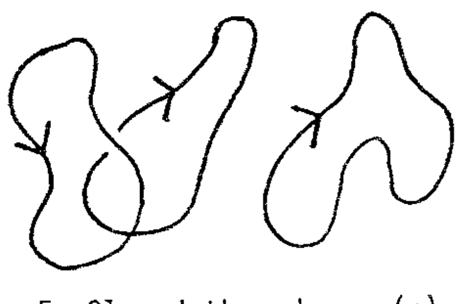

Fig. 5 Closed threads, $m_\mu(r)$

having spatial dimension R and temporal dimension T, then

$$V(R) = -\lim_{T\to\infty} \frac{1}{T} \ln \langle \exp[i \sum \phi(r,n)] \rangle \qquad (30)$$

Following the usual terminology of spin systems, if $\langle \exp[i \sum\phi(r,n)]\rangle$ is short-ranged in R (i.e. $V(R) \sim |R|$), then we say the gauge theory is "disordered", while if $\langle \exp[i \sum\phi(r,\hat{n})]\rangle$ is power-behaved (i.e. V(R) is short-ranged), then the theory is "ordered". The earlier discussion of the Abelian lattice gauge theory at finite temperature illustrates the general rationale behind these words.

Working from Eq. (29b) and inspired by the two dimensional XY model, a physical picture of the phases of Abelian lattice gauge theory results:

1. For $g < g_c$, the current loops $m_\mu(r)$ are small and irrelevant. The periodic character of the theory is irrelevant and it reduces to free electromagnetism. There is no quark confinement.
2. For $g > g_c$, the current loops are unbounded in size and are relevant. They disorder the system making the Wilson line integral correlation function fall at an exponential rate in R. Quark confinement follows.

This is an appealing physical picture. It also has many elements in common with some recent more general work by 't Hooft on quark confinement.[18] Unfortunately, the Partition function Eq. (29b) is difficult to use for quantitative calculations for several reasons. First, one must be able to enumerate closed current loops on the lattice. (It is possible to estimate the number of closed loops of a certain length from computer studies of constrained random walks.) Second, each element $m_\mu(r)$ of each loop interacts via a long range potential $V(r-r')$ with all the other elements $m_\mu(r')$. As in the lattice Coulomb gas problem, screening effects are essential in understanding the high temperature phase of this system.[17] Both of these problems have surfaced before in statistical mechanics--the counting problem appears in studies of dilute polymer solutions and the long range interactions between elements of closed threads appears in some models of the phases of ^{4}He.[19] One is led to consider constrained random walks in external fields which account for the long range interaction. The counting problem can then be written as a functional integral solution to the diffusion equation in an external field. M. Stone and P. Thomas[5] have carried out this program in an approximate fashion and have rewritten Eq. (29b) as scalar QED,

$$Z \sim \int d[A_\mu]d[\phi]d[\phi^*]\, e^{-\int [\frac{1}{4}F_{\mu\nu}^2 + (\nabla\phi)^2 + M^2\phi^2 + \lambda(\phi^*\phi)^2]d^4x} \qquad (31)$$

where M^2, the mass of the scalar field, is computed to be negative for large coupling g and positive for small g. This theory has two distinct phases; for $M^2 < 0$, a Higgs mechanism occurs and the vacuum supports the Meissner effect; for $M^2 > 0$, the vacuum is normal. If we were calculating the expectation value of the Wilson line integral in the Abelian lattice gauge theory, then after the duality trans-

transformations leading to the scalar QED formulation of the problem, we would be left with the calculation of the action of a pair of magmetic monopoles. But if the theory resides in the $M^2 < 0$ sector, the magnetic flux emanating from the poles forms flux tubes which generate a linear confinement potential. If $M^2 > 0$ there is no confinement. This is the correct picture of the phases of the theory, and the final reasoning is familiar, simple and reliable. It also bears out the intuitions of Mandelstam and 't Hooft who gave rough arguments mapping the problem of the confinement of colored quarks in QCD onto that of monopoles in a superconductor.[21] But Eq. (31) is not without difficulties. If the same approximations leading from Eq. (29b) to Eq. (31) are applied to the XY model in 3 dimensions one obtains scalar QED in 3 dimensions in place of Eq. (31). But this theory has a first order phase transition at its critical point while it is known from other analyses that the spin system has a second order transition. It would be useful to improve the arguments of ref. (5) and to pinpoint the source of the error.[21] Even though this approach is not quantitative in the immediate neighborhood of g_c, it is interesting and suggestive.

Using Eq. (31) we can consider lattice gauge theory at finite temperatures again. We must consider scalar QED in a heat bath. This problem has been studied in detail in other contexts and it was established that if the theory resided in the Higgs' phase (spontaneously broken symmetry) at low temperatures, then there will be a finite temperature T_c where the symmetry is restored.[22] In the language of the original lattice gauge theory this means that the theory will be quark confining up to a finite temperature T_c where the property will be abruptly lost. This is our earlier result now obtained from a different perspective.

CLOSING REMARKS

In all of the topics discussed in this review the concept of duality of statistical mechanics played an essential role. It is clear that this tool will be playing an increasingly important role in exact <u>and</u> approximate analyses of gauge theories. This powerful method, which predicted the critical point in the two dimensional Ising model years before Onsager solved it, will certainly be helpful in the quark confinement problem of QCD and, I believe, in our perception of gauge theories in general in the near future.

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